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ВТОРАЯ ЗАДАЧА ДАРБУ ДЛЯ УРАВНЕНИЯ ЭЙЛЕРА - ДАРБУ С ПАРАМЕТРАМИ $\alpha>0,\beta<0,-1<\alpha+\beta<1$

В работе построено общее решение уравнения Эйлера - Дарбу для параметров $\alpha>0,\beta<0,-1<\alpha+\beta<1$ **с помощью введения специальных функций и решена задача Дарбу.**

Рассмотрим уравнение

$$U_{\xi\eta}+\frac{\alpha}{\eta-\xi}U_{\xi}-\frac{\beta}{\eta-\xi}U_{\eta}=0$$

$$(\alpha>0,\beta<0,-1<\alpha+\beta<1) \qquad (1)$$

1. Построение общего решения путем введения специальных функций.

Пусть $z(\alpha,\beta)$ является решением уравнения (1). Тогда, используя общие свойства уравнения Эйлера - Дарбу, его решение можно представить в виде

$$z(\alpha,\beta)=\frac{\partial^m}{\partial\xi^m}z(\alpha-m,\beta)=$$

$$=\frac{\partial^m}{\partial\xi^m}(\eta-\xi)^{1-\alpha-\beta+m}z(1-\beta,1-\alpha+m)=$$

$$=\frac{\partial^m}{\partial\xi^m}(\eta-\xi)^{1-\alpha-\beta+m}$$

$$\frac{\partial^n}{\partial\xi^n}z(1-\beta-n,1-\alpha-m), \qquad (2)$$

где

$$z(1-\beta-n,1-\alpha+m)=$$

$$=x_1(\eta-\xi)^{\alpha+\beta+n-m-1}\int_0^1\varphi(\xi+(\eta-\xi)t)t^{n+\beta-1}(1-t)^{\alpha-m-1}dt+$$

$$+x_2\int_0^1\psi(\xi+(\eta-\xi)t)$$

$$t^{-\alpha+m}(1-t)^{-\beta-n}dt$$

$$(\text{здесь}\,0<\alpha-m<1,0<\beta+n<1,m,n\in N) \qquad (3)$$

Введем специальные функции

$$\varphi(\xi)\int_\xi^1 T(z)(z-\xi)^{\ell}dz,\psi(\xi)=\int_\xi^1 G(z)(z-\xi)^{\delta}dz, \qquad (4)$$

с учетом которых (3) примет вид

$$z(1-\beta-n,1-\alpha+m)=$$

$$=x_1\int_\xi^\eta(t-\xi)^{n+\beta-1}(\eta-t)^{\alpha-m-1}dt\int_t^1 T(z)(t-z)^{\ell}dz+$$

$$+x_2(\eta-\xi)^{\alpha+\beta+n-m-1}\int_\xi^\eta(t-\xi)^{-\alpha+m}(\eta-t)^{-\beta-n}dt$$

$$\int_t^1 G(z)(z-t)^{\delta}dz=x_1I_1+x_2I_2, \qquad (5)$$

где x_1I_1 и x_2I_2 – соответственно первое и второе слагаемые.

Изменяя пределы интегрирования, преобразуем I_1 к виду

$$I_1=\int_\xi^\eta T(z)dz\int_\xi^z(t-\xi)^{n+\beta-1}(\eta-t)^{\alpha-m-1}(z-t)^{\ell}dt+$$

$$+\int_\eta^1 T(z)dz\int_\xi^\eta(t-\xi)^{n+\beta-1}(\eta-t)^{\alpha-m-1}(z-t)^{\ell}dt=I_{11}+I_{12} \qquad (6)$$

В I_{11} сделаем замену переменной по формуле $\xi+(z-\xi)t=v$.

$$I_{11}=(\eta-\xi)^{\alpha-m-1}\int_\xi^\eta T(z)(z-\xi)^{n+\beta+\ell}dz\int_0^1 v^{n+\beta-1}(1-v)^{\ell}\cdot$$

$$\cdot(1-\frac{z-\xi}{\eta-\xi}v)^{\alpha-m-1}dv=\frac{\Gamma(n+\beta)\Gamma(\ell+1)}{\Gamma(n+\beta+\ell+1)}$$

$$(\eta-\xi)^{\alpha+\beta-n+m+\ell-1}\int_\xi^\eta T(z)(\frac{z-\xi}{\eta-\xi})^{n+\beta+\ell}F(1-\alpha+m,n+\beta;$$

$$n+\beta+\ell+1;\frac{z-\xi}{\eta-\xi})dz.$$

Проведем указанные в формуле (2) преобразования над I_{11}

$$\frac{\partial I_{11}}{\partial\xi}=\frac{\Gamma(n+\beta)\Gamma(\ell+1)}{\Gamma(n+\beta+\ell+1)}(\eta-\xi)^{\alpha+\beta+n-m+\ell-2}\int_\xi^\eta T(z)(\frac{z-\xi}{\eta-\xi})^{n+\beta+\ell-1}$$

$$\left[\frac{z-\xi}{\eta-\xi}\left(F(1-\alpha+m,n+\beta;n+\beta+\ell;\frac{z-\xi}{\eta-\xi})\right)\right.$$

$$(n+\beta+\ell)-(n+\alpha+\beta+\ell-m-1)F(1-\alpha+m,n+\beta;$$

$$n+\beta+\ell+1;\frac{z-\xi}{\eta-\xi}))-(n+\beta+\ell)F(1-\alpha+m,n+\beta;$$

$$\left.n+\beta+\ell;\frac{z-\xi}{\eta-\xi})\right]dz=\frac{\Gamma(n+\beta)\Gamma(\ell+1)}{\Gamma(n+\beta+\ell+1)}(n-\xi)^{\alpha+\beta+n-m+\ell-2}$$

$$\int_\xi^\eta T(z)(\frac{z-\xi}{\eta-\xi})^{n+\beta+\ell-1}\left[\frac{z-\xi}{\eta-\xi}(1-\alpha+m)F(2-\alpha+m,\right.$$

$$n+\beta;n+\beta+\ell+1;\frac{z-\xi}{\eta-\xi})-(n+\beta+\ell)F(1-\alpha+m,n+\beta;$$

$$\left.n+\beta+\ell;\frac{z-\xi}{n-\xi})\right]dz=\frac{\Gamma(n+\beta)\Gamma(\ell+1)}{\Gamma(n+\beta+\ell+1)}$$

$$(\eta-\xi)^{\alpha+\beta+n-m+\ell-2}\int_\xi^\eta T(z)\left(\frac{z-\xi}{\eta-\xi}\right)^{n+\beta+\ell-1}$$

$$\left(-(n+\beta+\ell)F(1-\alpha+m,n+\beta-1;n+\beta+\ell;\frac{z-\xi}{\eta-\xi})dz=\right.$$

$$-\frac{\Gamma(n+\beta)\Gamma(\ell+1)}{\Gamma(n+\beta+\ell)}(\eta-\xi)^{\alpha+\beta+n-m+\ell-2}\int_\xi^\eta T(z)(\frac{z-\xi}{\eta-\xi})^{n+\beta+\ell-1}$$

$$F(1-\alpha+m,n+\beta-1;n+\beta+\ell;\frac{z-\varsigma}{\eta-\xi})dz.$$

Вообще,

$$\frac{\partial^n}{\partial\xi^n}I_{11}=(-1)^n\frac{\Gamma(\beta+n)\Gamma(\ell+1)}{\Gamma(\beta+\ell+1)}(n-\xi)^{\alpha+\beta+\ell-m-1}\int_\xi^\eta T(z)(\frac{z-\xi}{\eta-\xi})^{\beta+\ell}$$

$$F\left(1-\alpha-m,\beta;\beta+\ell+1;\frac{z-\xi}{\eta-\xi}\right)dz$$

Полученный результат умножим на $(\eta-\xi)^{1-\alpha-\beta-m}$

$$(\eta-\xi)^{1-\alpha-\beta+m}\frac{\partial^n I_{11}}{\partial\xi^n}=(-1)^n\frac{\Gamma(\beta+n)\Gamma(\ell+1)}{\Gamma(\beta+\ell+1)}(\eta-\xi)^\ell\cdot$$

$$\int_\xi^\eta T(z)\left(\frac{z-\xi}{\eta-\xi}\right)^{\beta+\ell}F(1-\alpha+m,\beta;\beta+\ell+1;\frac{z-\xi}{\eta-\xi})dz$$

и найдем первую производную по переменной ξ

$$\frac{\partial}{\partial\xi}\left((\eta-\xi)^{1-\alpha-\beta+m}\frac{\partial^n I_{11}}{\partial\xi^n}\right)=$$

$$=(-1)^n\frac{\Gamma(\beta+n)\Gamma(\ell+1)}{\Gamma(\beta+\ell+1)}(\eta-\xi)^{\ell-1}\int_\xi^\eta T(z)\left(\frac{z-\xi}{\eta-\xi}\right)^{\beta+\ell-1}$$

$$\left[\frac{z-\xi}{\eta-\xi}\left(-\ell F(1-\alpha+m,\beta;\beta+\ell+1;\frac{z-\xi}{\eta-\xi})+\right.\right.$$

$$\left.+(\beta+\ell)F(1-\alpha+m,\beta;\beta+\ell;\frac{z-\xi}{\eta-\xi})\right)-(\beta+\ell)$$

$$F(1-\alpha+m,\beta;\beta+\ell;\frac{z-\xi}{\eta-\xi})\Big]dz=$$

$$=(-1)^n\frac{\Gamma(\beta+n)\Gamma(\ell+1)}{\Gamma(\beta+\ell+1)}(\eta-\xi)^{\ell-1}\int_\xi^\eta T(z)(\frac{z-\xi}{\eta-\xi})^{\beta+\ell-1}$$

$$\left(\frac{z-\xi}{\eta-\xi}\beta F(1-\alpha+m,1+\beta;\beta+\ell+1)-(\beta+\ell)F(1-\alpha+m,\beta;\right.$$

$$\left.\beta+\ell;\frac{z-\xi}{\eta-\xi})\right)dz=-(-1)^n\frac{\Gamma(\beta+n)\Gamma(\ell+1)}{\Gamma(\beta+\ell)}(\eta-\xi)^{\ell-1}$$

$$\int_\xi^\eta T(z)\left(\frac{z-\xi}{\eta-\xi}\right)^{\beta+\ell-1}F(m-\alpha,\beta;\beta+\ell;\frac{z-\xi}{\eta-\xi})dz$$

Вообще,

$$\frac{\partial^m}{\partial\xi^m}(-1)^{1-\alpha-\beta+m}\frac{\partial^n}{\partial\xi^n}I_{11}=$$

$$=(-1)^{m+n}\frac{\Gamma(\beta+n)\Gamma(\ell+1)}{\Gamma(\beta+\ell+1-m)}(\eta-\xi)^{\ell-m}\int_\xi^\eta T(z)\left(\frac{z-\xi}{\eta-\xi}\right)^{\beta+\ell-m}$$

$$F(1-\alpha,\beta;\beta+\ell+1-m;\frac{z-\xi}{\eta-\xi})dz \qquad (7)$$

Преобразуем второе слагаемое формулы (6)

$$I_{12}=\int_\eta^1 T(z)dz\int_\xi^\eta(t-\xi)^{n+\beta-1}(\eta-t)^{\alpha-m-1}(z-t)^\ell dt$$

С помощью подстановки $\xi+(z-\xi)t=v$ оно сводится к виду

$$I_{12}=(\eta-\xi)^{n+\alpha+\beta-m-1}\int_\eta^1 T((z)(z-\xi)^\ell dz\cdot$$

$$\int_0^1 v^{\beta+n-1}(1-v)^{\alpha-m-1}\left(1-\frac{\eta-\xi}{z-\xi}v\right)^\ell dv=$$

$$=\frac{\Gamma(\beta+n)\Gamma(\alpha-m)}{\Gamma(\alpha+\beta+n-m)}(\eta-\xi)^{\alpha+\beta+n-m+\ell-1}$$

$$\int_\eta^1 T(z)(\frac{\eta-\xi}{z-\xi})^{-\ell}F(-\ell,\beta+n;\alpha+\beta+n-m;\frac{\eta-\xi}{z-\xi})dz$$

Снова выполняем указанные в (2) преобразования

$$\frac{\partial I_{12}}{\partial\xi}=\frac{\Gamma(\beta+n)\Gamma(\alpha-m)}{\Gamma(\alpha+\beta+n-m)}(\eta-\xi)^{n+\alpha+\beta-m+\ell-2}$$

$$\int_\eta^1 T(z)(\frac{\eta-\xi}{z-\xi})^{-\ell}\Big[(-(\alpha+\beta+n-m+\ell-1)$$

$$F(-\ell,\beta+n;\beta+n+\alpha-m;\frac{\eta-\xi}{z-\xi})+$$

$$+\ell F(1-\ell,\beta+n;\alpha+\beta+n-m;\frac{\eta-\xi}{z-\xi}))-\ell\frac{\eta-\xi}{z-\xi}$$

$$F(1-\ell,\beta+n;\alpha+\beta+n-m;\frac{\eta-\xi}{z-\xi}\Big]dz=$$

$$=\frac{\Gamma(\beta+n)\Gamma(\alpha-m)}{\Gamma(\alpha+\beta+n-m)}(\eta-\xi)^{\alpha+\beta+n-m+\ell-2}\int_\eta^1 T(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{-\ell}$$

$$\left(-(\alpha+\beta+n-m-1)F(-\ell,\beta+n;\alpha+\beta+n-m-1;\frac{\eta-\xi}{z-\xi})-\right.$$

$$\left.-\ell\frac{\eta-\xi}{z-\xi}F(1-\ell,\beta+n;\alpha+\beta+n-m;\frac{\eta-\xi}{z-\xi}\right)dz=$$

$$-\frac{\Gamma(\beta+n)\Gamma(\alpha-m)}{\Gamma(\alpha+\beta+n-m-1)}(\eta-\xi)^{\alpha+\beta+n-m+\ell-2}\int_\eta^1 T(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{-\ell}$$

$$F(-\ell,\beta+n-1;\alpha+\beta+n-m-1;\frac{\eta-\xi}{z-\xi})dz.$$

Сравнивая I_{12} с $\frac{\partial}{\partial\xi}I_{12}$, можно заключить, что

$$\frac{\partial^n}{\partial\xi^n}I_{12}=\frac{\Gamma(\beta+n)\Gamma(\alpha-m)}{\Gamma(\alpha+\beta-m)}(-1)^n(\eta-\xi)^{\alpha+\beta-m+\ell-1}$$

$$\int_\eta^1 T(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{-\ell}F(-\ell,\beta;\alpha+\beta-m;\frac{\eta-\xi}{z-\xi})dz$$

Снова умножаем полученный результат на $(\eta-\xi)^{1-\alpha-\beta-m}$ и находим производную

$$\frac{\partial}{\partial\xi}((\eta-\xi)^{1-\alpha-\beta+m}\frac{\partial^n}{\partial\xi^n}I_{12})=(-1)^n\frac{\Gamma(\beta+n)\Gamma(\alpha-m)}{\Gamma(\alpha+\beta-m)}(\eta-\xi)^{\ell-1}$$

$$\int_\eta^1 T(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{-\ell}\frac{(-\ell)}{\alpha+\beta-m}$$

$$\Big(-\beta\frac{\eta-\beta}{z-\xi}F(1-\ell,1+\beta;\alpha+\beta-m+1;\frac{\eta-\xi}{z-\xi})+$$

$$+(\alpha+\beta-m)\frac{\eta-\xi}{z-\xi}F(1-\ell,\beta;\alpha+\beta-m;\frac{\eta-\xi}{z-\xi}))dz=$$

$$\frac{\Gamma(\alpha-m+1)\Gamma(\beta+n)}{\Gamma(\alpha+\beta-m+1)}(-\ell)(-1)^n(\eta-\xi)^{\ell-1}\int_\eta^1 T(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{-\ell+1}$$

$$F(1-\ell,\beta;\alpha+\beta-m+1;\frac{\eta-\xi}{z-\xi})dz$$

Вообще,

$$\frac{\partial^m}{\partial\xi^m}(\eta-\xi)^{1-\alpha-\beta-m}\frac{\partial^n I_{12}}{\partial\xi^n}=$$

$$=(-1)^n\frac{\Gamma(\alpha)\Gamma(\beta+n)}{\Gamma(\alpha+\beta)}(-\ell)(-\ell+1)...(-\ell+m-1)$$

$$\int_\eta^1 T(z)(z-\xi)^{\ell-m}F(m-\ell,\beta;\alpha+\beta;\frac{\eta-\xi}{z-\xi})dz=$$

$$(-1)^n\frac{\Gamma(\alpha)\Gamma(\beta+n)}{\Gamma(\alpha+\beta)}\frac{\Gamma(-\ell+m)}{\Gamma(-\ell)}\int_\eta^1 T(z)(z-\xi)^{\ell-m}$$

$$F(m-\ell,\beta;\alpha+\beta;\frac{\eta-\xi}{z-\xi})dz$$

Сведем выражение для I_{12}, входящее в формулу (5), к гипергеометрическим интегралам

$$I_2=(\eta-\xi)^{\alpha+\beta+n-m-1}\int_\xi^\eta(t-\xi)^{-\alpha+m}(\eta-t)^{-\beta-n}\int_t^1 G(z)(z-t)^\delta dz=$$

$$=(\eta-\xi)^{\alpha+\beta+n-m-1}\Big(\int_\xi^\eta G(z)dz\int_\xi^z(t-\xi)^{-\alpha+m}(\eta-t)^{-\beta-n}$$

$$(z-t)^\delta dt+\int_\eta^1 G(z)dz\int_\xi^\eta(t-\xi)^{-\alpha+m}(\eta-t)^{-\beta-n}(z-t)^\delta dt\Big)=$$

$$=I_{21}+I_{22}, \qquad (9)$$

где

$$I_{21}=(\eta-\xi)^{\alpha-m-1}\int_\xi^\eta G(z)(z-\xi)^{1-\alpha+m+\delta}dz$$

$$\int_0^1 v^{-\alpha+m}(1-v)^\delta\left(1-\frac{z-\xi}{\eta-\xi}v\right)^{-\beta-n}dv=$$

$$=\frac{\Gamma(1-\alpha+m)\Gamma(\delta+1)}{\Gamma(2-\alpha+m+\delta)}-(\eta-\xi)^\delta$$

$$\int_\xi^\eta G(z)(\frac{z-\xi}{\eta-\xi})^{1-\alpha+m+\delta}F(1-\alpha+m,\beta+n;$$

$$2-\alpha+m+\delta;\frac{z-\xi}{\eta-\xi})dz.$$

Выполняя преобразования, аналогичные приведенным выше, получим

$$\frac{\partial}{\partial\xi}I_{21}=\frac{\Gamma(1-\alpha+m)\Gamma(\delta+1)}{\Gamma(2-\alpha+m+\delta)}(\eta-\xi)^{\delta-1}\int_\xi^\eta G(z)(\frac{z-\xi}{\eta-\xi})^{-\alpha+m+\delta}\cdot$$

$$\cdot\Big[\frac{z-\xi}{\eta-\xi}(-\delta F(1-\alpha+m,\beta+n;2-\alpha+m+\delta;$$

$$\frac{z-\xi}{\eta-\xi})+(1-\alpha+m+\delta)F(1-\alpha+m,\beta+n;1-\alpha+$$

$$+m+\delta;\frac{z-\xi}{\eta-\xi}))-(1-\alpha+m+\delta)F(1-\alpha+m,\beta+n;$$

$$1-\alpha+m+\delta;\frac{z-\xi}{\eta-\xi})\Big]dz=\frac{\Gamma(1-\alpha+m)\Gamma(\delta+1)}{\Gamma(2-\alpha+m+\delta)}(\eta-\xi)^{\delta-1}\cdot$$

$$\cdot\int_\xi^\eta G(z)(\frac{z-\xi}{\eta-\xi})^{-\alpha+m+\delta}\Big[\frac{z-\xi}{\eta-\xi}(1-\alpha+m)\cdot$$

$$\cdot F(2-\alpha+m,\beta+n;2-\alpha+m+\delta;\frac{z-\xi}{\eta-\xi})-$$

$$-(1-\alpha+m+\delta)F(1-\alpha+m;\beta+n;1-\alpha+m+\delta;\frac{z-\xi}{\eta-\xi})\Big]dz=$$

$$-\frac{\Gamma(1-\alpha+m)\Gamma(\delta+1)}{\Gamma(1-\alpha+m+\delta)}(\eta-\xi)^{\delta-1}\int_\xi^\eta G(z)\left(\frac{z-\xi}{\eta-\xi}\right)^{-\alpha+m+\delta}\cdot$$

$$\cdot F(1-\alpha+m,\beta+n-1;1-\alpha+m+\delta;\frac{z-\xi}{\eta-\xi})dz.$$

Следовательно,

$$\frac{\partial^n I_{21}}{\partial\xi^n}=(-1)^n\frac{\Gamma(1-\alpha+m)\Gamma(\delta+1)}{\Gamma(2-\alpha+m+\delta-n)}(\eta-\xi)^{\delta-n}$$

$$\int_\xi^\eta G(z)\left(\frac{z-\xi}{\eta-\xi}\right)^{1-\alpha+m-n+\delta}\cdot$$

$$\cdot F(1-\alpha+m,\beta;2-\alpha+m-n+\delta;\frac{z-\xi}{\eta-\xi})dz.$$

Умножим полученный результат на $(\eta-\xi)^{1-\alpha-\beta+m}$ и найдем производную по переменной ξ.

$$\frac{\partial}{\partial\xi}(\eta-\xi)^{1-\alpha-\beta+m}\frac{\partial^n}{\partial\xi^n}I_{21}=(-1)^n\frac{\Gamma(1-\alpha+m)\Gamma(\delta+1)}{\Gamma(2-\alpha+m+\delta-n)}\frac{\partial}{\partial\xi}\cdot$$

$$\cdot((\eta-\xi)^{1-\alpha-\beta+m+\delta-n}\int_\xi^\eta G(z)\left(\frac{z-\xi}{\eta-\xi}\right)^{1-\alpha+m-n+\delta}\cdot$$

$$\cdot F(1-\alpha+m,\beta;2-\alpha+m-n+\delta;\frac{z-\xi}{\eta-\xi})dz)=$$

$$=(-1)^n\frac{\Gamma(1-\alpha+m)\Gamma(\delta+1)}{\Gamma(2-\alpha+m+\delta-n)}(\eta-\xi)^{-\alpha-\beta+m+\delta-n}\int_\xi^\eta G(z)\left(\frac{z-\xi}{\eta-\xi}\right)^{-\alpha+m-n+\delta}$$

$$\left[\frac{z-\xi}{\eta-\xi}(-(1-\alpha-\beta+m-n+\delta)F(1-\alpha+m,\beta;\right.$$

$$2-\alpha+m-n+\delta;\frac{z-\xi}{\eta-\xi}))-(1-\alpha+m-n+\delta)\cdot$$

$$\left.\cdot F(1-\alpha+m,\beta;1-\alpha+m-n+\delta;\frac{z-\xi}{\eta-\xi})\right]dz=$$

$$=(-1)^n\frac{\Gamma(1-\alpha+m)\Gamma(\delta+1)}{\Gamma(2-\alpha+m+\delta-n)}(\eta-\xi)^{-\alpha-\beta+m+\delta-n}\int_\xi^\eta G(z)\left(\frac{z-\xi}{\eta-\xi}\right)^{-\alpha+m-n+\delta}\cdot$$

$$\cdot\left(\frac{z-\xi}{\eta-\xi}\beta F(1-\alpha+m,1+\beta;2-\alpha+m-n+\delta;\frac{z-\xi}{\eta-\xi}\right)-$$

$$\left.-(1-\alpha+m-n+\delta)F(1-\alpha+m;\beta;1-\alpha+m-n+\delta;\frac{z-\xi}{\eta-\xi})\right)dz=$$

$$=(-1)^{n+1}\frac{\Gamma(1-\alpha+m)\Gamma(\delta+1)}{\Gamma(1-\alpha+m+\delta-n)}(\eta-\xi)^{-\alpha-\beta+m+\delta-n}\int_\xi^\eta G(z)\left(\frac{z-\xi}{\eta-\xi}\right)^{-\alpha+m-n+\delta}\cdot$$

$$\cdot F(-\alpha+m,\beta;1-\alpha+m-n+\delta;\frac{z-\xi}{\eta-\xi})dz.$$

Таким образом,

$$\frac{\partial^m}{\partial\xi^m}(\eta-\xi)^{1-\alpha-\beta+m}\frac{\partial^n}{\partial\xi^n}I_{21}=$$

$$=(-1)^{n+m}\frac{\Gamma(1-\alpha+m)\Gamma(\delta+1)}{\Gamma(2-\alpha+\delta-n)}(\eta-\xi)^{1-\alpha-\beta+\delta-n}\int_\xi^\eta G(z)\left(\frac{z-\xi}{\eta-\xi}\right)^{1-\alpha-n+\delta}$$

$$F(1-\alpha,\beta;2-\alpha-n+\delta;\frac{z-\xi}{\eta-\xi})dz \qquad (10)$$

Выполняя замену переменной по формуле $\xi+(\eta-\xi)v=t$, преобразуем I_{22} к виду

$$I_{22}=(\eta-\xi)^{\alpha+\beta+n-m-1}\int_\eta^1 G(z)dz\int_\xi^\eta(t-\xi)^{-\alpha+m}(\eta-t)^{-\beta-n}(z-t)^\delta dt=$$

$$=\int_\eta^1 G(z)(z-\xi)^\delta dz\int_0^1 v^{-\alpha+m}(1-v)^{-\beta-n}(1-\frac{\eta-\xi}{z-\xi}v)^\delta dv=$$

$$=\frac{\Gamma(1-\alpha+m)\Gamma(1-\beta-n)}{\Gamma(2-\alpha-\beta+m-n)}(\eta-\xi)^\delta\int_\eta^1 G(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{-\delta}\cdot$$

$$\cdot F(1-\alpha+m,-\delta;2-\alpha-\beta+m-n;\frac{\eta-\xi}{z-\xi})dz\cdot$$

$$\cdot\frac{\partial I_{22}}{\partial\xi}=\frac{\Gamma(1-\alpha+m)\Gamma(1-\beta-n)}{\Gamma(2-\alpha-\beta+m-n)}(-\delta)(\eta-\xi)^{\delta-1}\int_\eta^1 G(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{-\delta}\cdot$$

$$\cdot\left[(F(1-\alpha+m,-\delta;2-\alpha-\beta+m-n;\frac{\eta-\xi}{z-\xi})-\right.$$

$$-F(1-\alpha+m,1-\delta;2-\alpha-\beta+m-n;\frac{\eta-\xi}{z-\xi}))+$$

$$\left.+\frac{\eta-\xi}{z-\xi}F(1-\alpha+m,1-\delta;2-\alpha-\beta+m-n;\frac{\eta-\xi}{z-\xi})\right]dz=$$

$$=\frac{\Gamma(1-\alpha+m)\Gamma(1-\beta-n)}{\Gamma(2-\alpha-\beta+m-n)}(-\delta)(\eta-\xi)^{\delta-1}\int_\eta^1 G(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{-\delta}$$

$$\frac{1}{2-\alpha-\beta+m-n}\left(-(1-\alpha+m)\frac{\eta-\xi}{z-\xi}F(2-\alpha+m,1-\delta;\right.$$

$$3-\alpha-\beta+m-n;\frac{\eta-\xi}{z-\xi})+\frac{\eta-\xi}{z-\xi}(2-\alpha-\beta+m-n)$$

$$F(1-\alpha+m,1-\delta;2-\alpha-\beta+m-n;\frac{\eta-\xi}{z-\xi}))dz=$$

$$=\frac{\Gamma(1-\alpha+m)\Gamma(2-\beta-n)}{\Gamma(3-\alpha-\beta+m-n)}(-\delta)(\eta-\xi)^{\delta-1}\int_\eta^1 G(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{1-\delta}$$

$$F(1-\alpha+m,1-\delta;3-\alpha-\beta+m-n;\frac{\eta-\xi}{z-\xi})dz$$

Следовательно,

$$\frac{\partial^n I_{22}}{\partial\xi^n}=\frac{\Gamma(1-\alpha+m)\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta+m)}(-\delta)(1-\delta)\ldots\ldots$$

$$(n-1-\delta)(\eta-\xi)^{\delta-n}\int_\eta^1 G(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{n-\delta}$$

$$F(1-\alpha+m,n-\delta;2-\alpha-\beta+m;$$

$$\frac{\eta-\xi}{z-\xi})dz=(-1)^n\frac{\Gamma(1-\alpha+m)\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta+m)}\frac{\Gamma(\delta+1)}{\Gamma(\delta-n+1)}$$

$$(\eta-\xi)^{\delta-n}\int_\eta^1 G(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{n-\delta}$$

$$F(1-\alpha+m,n-\delta;2-\alpha-\beta+m;$$

$$\frac{\eta-\xi}{z-\xi})dz$$

Умножим полученный результат на $(\eta-\xi)^{1-\alpha-\beta-m}$, найдем производную по переменной ξ.

$$\frac{\partial}{\partial\xi}(\eta-\xi)^{1-\alpha-\beta+m-n+\delta}\frac{\Gamma(1-\alpha-m)\Gamma(1-\beta)\Gamma(n-\delta)}{\Gamma(2-\alpha-\beta+m)\Gamma(-\delta)}$$

$$\int_\eta^1 G(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{n-\delta}F(1-\alpha+m,n-\delta;2-\alpha-\beta+m;\frac{\eta-\xi}{z-\xi})dz=$$

$$(-1)^n \frac{\Gamma(1-\alpha+m)\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta+m)} \frac{\Gamma(\delta+1)}{\Gamma(\delta-n+1)}(\eta-\xi)^{-\alpha-\beta+m-n+\delta}$$

$$\int_\eta^1 G(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{n-\delta}[(-(1-\alpha-\beta+m-n+\delta)F(1-\alpha+m,$$

$$n-\delta;2-\alpha-\beta+m;\frac{\eta-\xi}{z-\xi})-(\eta-\delta)$$

$$F(1-\alpha+m,n-\delta+1;2-\alpha-\beta+m;\frac{\eta-\xi}{z-\xi}))+$$

$$+(n-\delta)\frac{\eta-\xi}{z-\xi}F(1-\alpha+m,n-\delta+1;$$

$$2-\alpha-\beta+m;\frac{\eta-\xi}{z-\xi})]dz=$$

$$=(-1)^n \frac{\Gamma(1-\alpha+m)\Gamma(1-\beta)\Gamma(1+\delta)}{\Gamma(2-\alpha-\beta+m)\Gamma(1+\delta-n)}(\eta-\xi)^{-\alpha-\beta+m-n+\delta}$$

$$\int_\eta^1 G(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{n-\delta}[-(1-\alpha-\beta+m)$$

$$F(1-\alpha+m,n-\delta;1-\alpha-\beta+m;\frac{\eta-\xi}{z-\xi})+$$

$$+(n-\delta)F(1-\alpha+m,n-\delta+1,2-\alpha-\beta+m;\frac{\eta-\xi}{z-\xi})]dz=$$

$$-\frac{\Gamma(1-\alpha+m)\Gamma(1-\beta)\Gamma(1+\delta)}{\Gamma(1-\alpha-\beta+m)\Gamma(1+\delta-n)}(\eta-\xi)^{-\alpha-\beta+m-n+\delta}(-1)^n$$

$$\int_\eta^1 G(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{n-\delta}F(-\alpha+m,n-\delta;1-\alpha-\beta+m;\frac{\eta-\xi}{z-\xi})dz$$

Итак,

$$\frac{\partial^m}{\partial\xi^m}(\eta-\xi)^{1-\alpha-\beta+m}\frac{\partial^n}{\partial\xi^n}I_{22}=$$

$$=(-1)^{m+n}\frac{\Gamma(1-\alpha+m)\Gamma(1-\beta)\Gamma(1+\delta)}{\Gamma(1+\delta-n)\Gamma(2-\alpha-\beta)}$$

$$(\eta-\xi)^{1-\alpha-\beta-n+\delta}\int_\eta^1 G(z)\left(\frac{\eta-\xi}{z-\xi}\right)^{n-\delta}$$

$$F(1-\alpha,n-\delta;2-\alpha-\beta;\frac{\eta-\xi}{z-\xi})dz. \qquad (11)$$

Подставим (7), (8), (10), (11) в (2), тогда с учетом (5), (6) и (9) получим общее решение уравнения (1).

$$U(\xi,\eta)=x_1(-1)^{m+n}\frac{\Gamma(\beta+n)\Gamma(\ell+1)}{\Gamma(\beta+\ell+1-m)}(\eta-\xi)^{-\beta}$$

$$\int_\xi^\eta T(z)(z-\xi)^{\beta+\ell-m}F(1-\alpha,\beta;\beta+\ell+1-m;\frac{z-\xi}{\eta-\xi})dz+x_1(-1)^n$$

$$\frac{\Gamma(\alpha)\Gamma(\beta+n)\Gamma(-\ell+m)}{\Gamma(\alpha+\beta)\Gamma(-\ell)}\int_\eta^1 T(z)(z-\xi)^{\ell-m}$$

$$F(m-\ell,\beta,\alpha+\beta;\frac{\eta-\xi}{z-\xi})dz+x_2(-1)^{m+n}$$

$$\frac{\Gamma(1-\alpha+m)\Gamma(\delta+1)}{\Gamma(2-\alpha+\delta-n)}(\eta-\xi)^{-\beta}\int_\xi^\eta G(z)(z-\xi)^{1-\alpha-n+\delta}$$

$$F(1-\alpha,\beta;2-\alpha-n+\delta;\frac{z-\xi}{\eta-\xi})dz+(\eta-\xi)^{1-\alpha-\beta}x_2(-1)^{m+n}$$

$$\frac{\Gamma(1-\alpha+m)\Gamma(1-\beta)\Gamma(1+\delta)}{\Gamma(1+\delta-n)\Gamma(2-\alpha-\beta)}\int_\eta^1 G(z)(z-\xi)^{\delta-n}$$

$$-1<\alpha+\beta<1$$

$$F(1-\alpha,n-\delta;2-\alpha-\beta;\frac{\eta-\xi}{z-\xi})dz. \qquad (12)$$

Обозначим $\ell-m=\mu,-\beta-1<\mu<-\beta;$ в частности за μ можно взять число

$$\mu=-\alpha-\beta+m$$

$$(-\beta-1<-\alpha-\beta+m<-\beta)$$

Параметр δ выберем, исходя из того, что

$$\delta>-2+\alpha+n \quad (\alpha>1)$$

В частности за δ можно взять

$$\delta=m+n-1$$

Произвольные постоянные x_1 и x_2 полагаем равными

$$x_1=(-1)^n\frac{\Gamma(\alpha+\beta)\Gamma(-\ell)}{\Gamma(\alpha)\Gamma(\beta+n)\Gamma(-\ell+m)}=$$

$$=\frac{(-1)^n\Gamma(\alpha+\beta)\Gamma(\alpha+\beta-2m)}{\Gamma(\alpha)\Gamma(\beta+n)\Gamma(\alpha+\beta-m)},$$

$$x_2=(-1)^{m+n}\frac{\Gamma(1+\delta-n)\Gamma(2-\alpha-\beta)}{\Gamma(1-\alpha+m)\Gamma(1-\beta)\Gamma(1+\delta)}=$$

$$=(-1)^{n+m}\frac{(m-1)!\Gamma(2-\alpha-\beta)}{\Gamma(1-\alpha+m)\Gamma(1-\beta)(m+n-1)!}.$$

С учетом выбора μ,δ,x_1 и x_2 решение уравнения (1) примет вид

$$U(\xi,\eta)=\frac{\Gamma(\alpha+\beta)\Gamma(1-\alpha-\beta+m)}{\Gamma(\alpha)\Gamma(1-\alpha+m)}(\eta-\xi)^{-\beta}$$

$$\int_\xi^\eta T(z)(z-\xi)^{-\alpha+m}F(1-\alpha;\beta;1-\alpha+m;\frac{z-\xi}{\eta-\xi})dz+$$

$$\int_\eta^1 T(z)(z-\xi)^{-\alpha-\beta+m}F(\alpha+\beta-m,\beta;\alpha+\beta;\frac{\eta-\xi}{z-\xi})dz+$$

$$+\frac{(m-1)!\Gamma(2-\alpha-\beta)}{(m+n-1)!\Gamma(1-\alpha+m)\Gamma(1-\beta)}(\eta-\xi)^{-\beta}\int_\xi^\eta G(z)(z-\xi)^{m-\alpha}$$

$$F(1-\alpha,\beta;1+m-\alpha;\frac{z-\xi}{\eta-\xi})dz+(\eta-\xi)^{1-\alpha-\beta}$$

$$\int_\eta^1 G(z)(z-\xi)^{m-1}F(1-\alpha,-m+1;2-\alpha-\beta;\frac{\eta-\xi}{z-\xi})dz. \qquad (13)$$

2. Постановка и решение задачи Дарбу

Задача. В области D, ограниченной линиями $\xi=0,\eta=1,\eta=\xi(I)$, найти решение $U(\xi,\eta)\in C^{(n)}(\overline{D})\cup C^{(n-1)}(D\cup I)$ уравнения (1), удовлетворяющее краевым условиям

$$U(\xi,\xi)=\tau(\xi)=\int_{\xi}^{1}T(z)(z-\xi)^{-\alpha-\beta+m}dz \qquad (14)$$

$$U(\xi,1)=\Psi(\xi)=I_{\xi 1}^{1-\alpha+m,\alpha-m-1+\beta,\alpha-1}X(\xi) \qquad (15)$$

Из (13) сразу следует, что общее решение уравнения (1) удовлетворяет краевому условию (14).

На характеристике $\eta=1$ решение уравнения (1) примет вид

$$U(\xi,1)=\frac{\Gamma(\alpha+\beta)\Gamma(1-\alpha-\beta+m)}{\Gamma(\alpha)}\frac{(1-\xi)^{-\beta}}{\Gamma(1-\alpha+m)}$$

$$\int_{\xi}^{1}T(z)(z-\xi)^{m-\alpha}F(1-\alpha,\beta,1-\alpha+m;\frac{z-\xi}{1-\xi})dz+$$

$$\frac{(m-1)!\Gamma(2-\alpha-\beta)}{(m+n-1)!\Gamma(1-\beta)}\frac{(1-\xi)^{-\beta}}{\Gamma(1-\alpha+m)}$$

$$\int_{\xi}^{1}G(z)(z-\xi)^{m-\alpha}F(1-\alpha,\beta;1-\alpha+m;\frac{z-\xi}{1-\xi})dz=$$

$$=\frac{\Gamma(\alpha+\beta)\Gamma(1-\alpha-\beta+m)}{\Gamma(\alpha)}$$

$$I_{\xi 1}^{1-\alpha+m,\alpha+\beta-m-1,\alpha-1}T(\xi)+$$

$$+\frac{(m-1)!\Gamma(2-\alpha-\beta)}{(m+n-1)!\Gamma(1-\beta)}I_{\xi 1}^{1-\alpha+m,\alpha+\beta-m-1,\alpha-1}G(\xi), \qquad (16)$$

где $I_{\xi 1}^{1-\alpha+m,\alpha+\beta-m-1,\alpha-1}f$ – оператор *Saigo.*

Оператор обращения имеет вид $I_{\xi 1}^{\alpha-m-1,-\alpha-\beta+m+1,m}$, т. е.

$$I_{\xi 1}^{\alpha-m-1,-\alpha-\beta+m+1,m}U(\xi,1)=\frac{\Gamma(\alpha+\beta)\Gamma(1-\alpha-\beta+m)}{\Gamma(\alpha)}T(\xi)+$$

$$+\frac{(m-1)!\Gamma(2-\alpha-\beta)}{(m+n-1)!\Gamma(1-\beta)}G(\xi). \qquad (17)$$

Учитывая, что $U(\xi,1)=I_{\xi 1}^{1-\alpha+m,\alpha-m-1+\beta,\alpha-1}X(\xi)$, получим

$$-\frac{d}{d\xi}I_{\xi 1}^{\alpha-m,-\alpha-\beta+m,m-1}I_{\xi 1}^{1-\alpha+m,\alpha-m-1+\beta,\alpha-1}X(\xi)=$$

$$=-\frac{d}{d\xi}I_{\xi 1}^{1,-1,m-1}X(\xi)=-\frac{d}{d\xi}\int_{\xi}^{1}X(t)dt=X(\xi).$$

Подставляя этот результат в (17), получим основное соотношение, из которого определим функцию

$$G(\xi)=\frac{\Gamma(1-\beta)(m+n-1)!}{(m-1)!\Gamma(2-\alpha-\beta)}.$$

$$\cdot\left(X(\xi)-\frac{\Gamma(\alpha+\beta)\Gamma(1-\alpha-\beta+m)}{\Gamma(\alpha)}T(\xi)\right)$$

Список использованной литературы:
1. Градштейн И.С., Рыжик И.М. Таблицы интегралов, сумм рядов и произведений.: Физматгиз, 1962. 1100 с.
2. Hardy G., Liltlewood I., Some propertiesof fractional integraes. I Math Z., 27., 565-606. 1928.
3. Saigo M., Math Rep. Kyushu Univ, 1978, Vol 11.
4. Saigo M., Math., Jap., 1979. Vol 24.